Of Antipodes and Involutions, Of Cabbages and Kings

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Hopf algebras and antipodes

Sign reversing involutions

Other work and open problems

The introductory material is intended to be intuitive if not completely rigorous. All vector spaces will be over a field $\mathbb{F}$ and all maps on them will be linear. An *algebra*, H, is a vector space over $\mathbb{F}$ together with the following.

(1) An associative multiplication of elements of H. This can be viewed as a map $m : H \otimes H \to H$ by

$$g \otimes h \mapsto g \cdot h$$

making a certain diagram commute.

(2) A unit element. This can be viewed as a map $u : \mathbb{F} \to H$ by

$$1_{\mathbb{F}} \mapsto 1_H$$

making a certain diagram commute.

Note that since multiplication is associative we have a well-defined map $m^{k-1} : H^{\otimes k} \to H$ by

$$h_1 \otimes h_2 \otimes \cdots \otimes h_k \mapsto h_1 \cdot h_2 \cdot \ \ldots \ \cdot h_k.$$

Example: The shuffle algebra. Let A be a finite set called the *alphabet*. Consider the *words over A*

$$A^* = \{w = a_1 a_2 \ldots a_n \ : \ a_i \in A \text{ for all } i, \text{ and } n \geq 0\}.$$

The *shuffle algebra* is the set of finite formal sums

$$\mathbb{F}A^* = \left\{ \sum_w c_w w \ : \ c_w \in \mathbb{F} \text{ for each } w \in A^* \text{ in the sum} \right\}.$$

The *shuffles* of $u, v \in A^*$ are the elements of the multiset $u \sqcup\!\sqcup\, v$ of all interleavings of u and v. Different interleavings are considered distinct even if they result in the same word. For example

$$a \sqcup\!\sqcup\, ab = \{\{aab, aab, aba\}\} \implies a \cdot ab = 2aab + aba.$$

The multiplication in $\mathbb{F}A^*$ is by shuffling

$$u \cdot v = \sum_{w \in u \sqcup\!\sqcup v} w.$$

The identity element is the empty word e since, for any $v \in A^*$,

$$e \sqcup\!\sqcup\, v = v \sqcup\!\sqcup\, e = \{\{v\}\}.$$

A *coalgebra*, H, is a vector space over $\mathbb{F}$ together with the following.

(1) A comultiplication of elements of H which is a map $\Delta : H \to H \otimes H$ written as

$$\Delta(h) = \sum h_1 \otimes h_2.$$

Comultiplication is assumed to be coassociative in that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

where id is the identity. This can be expressed using the dual of the commutative diagram for associativity of multiplication.

(2) A counit element which is a map $\epsilon : H \to \mathbb{F}$ making the dual of the diagram for a unit commute.

Note that since comultiplication is coassociative we have a well-defined map $\Delta^{k-1} : H \to H^{\otimes k}$ by

$$h \mapsto \sum h_1 \otimes h_2 \otimes \cdots \otimes h_k.$$

Example (continued). The shuffle algebra is also a coalgebra. The comultiplication is, for $w \in A^*$,

$$\Delta(w) = \sum_{w_1 w_2 = w} w_1 \otimes w_2$$

where $w_1 w_2$ is concatenation. To illustrate

$$\Delta(aab) = e \otimes aab + a \otimes ab + aa \otimes b + aab \otimes e.$$

The counit is, for $w \in A^*$,

$$\epsilon(w) = \begin{cases} 1 & \text{if } w = e, \\ 0 & \text{else.} \end{cases}$$

The comultiplication is coassociative with

$$\Delta^2(w) = \sum_{w_1 w_2 w_3 = w} w_1 \otimes w_2 \otimes w_3.$$

A *bialgebra*, H, is a vector space over $\mathbb{F}$ which is both an algebra and a coalgebra such that the maps Δ and ϵ are algebra homomorphisms. Suppose H has a vector space decomposition

$$H = \bigoplus_{n \geq 0} H_n.$$

If $h \in H_n$ then we say h is *homogeneous of degree n* and write $\deg h = n$.

Call bialgebra H *graded* if the direct sum satisfies the following.

(1) If $\deg g = m$ and $\deg h = n$ then

$$\deg(g \cdot h) = m + n.$$

(2) If $\deg h = n$ and $\Delta h = \sum h_1 \otimes h_2$ then, for all h_1 and h_2,

$$\deg h_1 + \deg h_2 = n.$$

Call a graded bialgebra *connected* if $H_0 \cong \mathbb{F}$.

Example (continued). Say $w = a_1 \dots a_n \in A^*$ has *length* $|w| = n$. Let

$$A^n = \{w \in A^* \ : \ |w| = n\}.$$

Then

$$\mathbb{F}A^* = \bigoplus_{n \geq 0} \mathbb{F}A^n.$$

This makes $\mathbb{F}A^*$ graded:

(1) We have $u \cdot v = \sum_{w \in u \sqcup\!\sqcup v} w$ and

$$w \in u \sqcup\!\sqcup v \implies |w| = |u| + |v|.$$

(2) We have $\Delta(w) = \sum_{w_1 w_2 = w} w_1 \otimes w_2$ and

$$w_1 w_2 = w \implies |w_1| + |w_2| = |w|.$$

We also have that $\mathbb{F}A^*$ is connected since

$$\mathbb{F}A^0 = \mathbb{F}\{e\} \cong \mathbb{F}.$$

A *Hopf algebra*, H, is a bialgebra together with a map $S : H \to H$ called the *antipode* making a certain diagram commute.

(a) Every group G gives rise to a Hopf algebra with, for all $g \in G$,

$$S(g) = g^{-1}.$$

(b) In the Hopf algebra of symmetric functions we have

$$S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda^t},$$

where s_λ is a Schur function, $|\lambda|$ is the sum of the parts of the partition λ, and λ^t is its transpose.

Theorem (Takeuchi)

Let H be a connected, graded bialgebra. Then H is a Hopf algebra with, for $h \in H_n$,

$$S(h) = \sum_{k=1}^{n} (-1)^k \sum_{h_1,\ldots,h_k} h_1 \cdot \ \ldots \ \cdot h_k$$

where $h_1 \otimes \cdots \otimes h_k$ is a term in $\Delta^{k-1}(h)$ and $\deg h_i \geq 1$ for all i.

$$S(h) = \sum_{k=1}^{n} (-1)^k \sum_{h_1,\dots,h_k} h_1 \cdot \ \dots \ \cdot h_k$$

where $h_1 \otimes \dots \otimes h_k$ is a term in $\Delta^{k-1}(h)$ and $\deg h_i \geq 1$ for all i.

Example (continued). If $w = ab$ then $|w| = 2$ and so by Takeuchi

$$\begin{aligned} S(ab) &= \sum_{k=1}^{2} (-1)^k \sum_{w_1 \dots w_k = w} w_1 \sqcup\!\sqcup \dots \sqcup\!\sqcup w_k \\ &= (-1)^1 (ab) + (-1)^2 (a \sqcup\!\sqcup b) \\ &= -ab + (ab + ba) \\ &= ba. \end{aligned}$$

Define the *reversal* of $w = a_1 \dots a_n$ to be $\operatorname{rev} w = a_n \dots a_1$.

Theorem

If $w \in A^*$ has $|w| = n$ then

$$S(w) = (-1)^n \operatorname{rev} w.$$

Let X be a finite set which is *signed* in that there is a function

$$\operatorname{sgn} : X \to \{+1, -1\}.$$

An involution ι on X is *sign reversing* if, for every 2-cycle (x, y) of ι,

$$\operatorname{sgn} y = -\operatorname{sgn} x.$$

It follows that

$$\sum_{x \in X} \operatorname{sgn} x = \sum_{x \in \operatorname{fix} \iota} \operatorname{sgn} x$$

where $\operatorname{fix} \iota$ is the set of fixed points of ι. Given X, one tries to construct ι so that the second sum has fewer terms and may even be cancellation free.

Theorem

If $w \in A^*$ has $|w| = n$ then: $\quad S(w) = (-1)^n \operatorname{rev} w.$

Proof. By Takeuchi

$$S(w) = \sum_{k=1}^{n} (-1)^k \sum w_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup w_k.$$

The inner sum is over all $w_1 \ldots w_k = w$ with $|w_i| \geq 1$ for all i. Let

$X = \{\ x = (w_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup w_k, v) \mid v \text{ is a term in } w_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup w_k\ \},$

$\operatorname{sgn}(w_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup w_k, v) = (-1)^k.$

Thus

$$S(w) = \sum_{x \in X} (\operatorname{sgn} x) v.$$

Example (continued). If $w = ab$ then

$$S(ab) = (-1)^1(ab) + (-1)^2(a \sqcup\!\sqcup b) = -ab + (ab + ba).$$

So

$$X = \{(ab, ab),\ (a \sqcup\!\sqcup b, ab),\ (a \sqcup\!\sqcup b, ba)\}.$$

Write $\omega = w_1 \sqcup \ldots \sqcup w_k$. Given (ω, v) we say w_i is *splittable* if

$$|w_i| \geq 2.$$

Say $w_i = ab \ldots c$. In this case we can apply the *splitting map*

$$\sigma(\omega, v) = (\omega', v)$$

where ω' is obtained from ω by replacing w_i by

$$w_i' \sqcup w_{i+1}' = a \sqcup b \ldots c.$$

Note that $(\omega', v) \in X$ since v is still a term in ω'.
Ex. Suppose $w = abcdefg$ and

$$(\omega, v) = (a \sqcup b \sqcup cde \sqcup fg, fcbdgea).$$

Then $w_3 = cde$ is splittable and

$$\sigma(\omega, v) = (a \sqcup b \sqcup c \sqcup de \sqcup fg, fcbdgea).$$

Given (ω, v) we say w_i is *mergeable* with w_{i+1} if

$$|w_i| = 1 \text{ and } w_i \text{ is to the left of } w_{i+1} \text{ in } v.$$

Say $w_i = a$ and $w_{i+1} = b \dots c$. In this case we can apply the *merging map*

$$\mu(\omega, v) = (\omega', v)$$

where ω' is obtained from ω by replacing $w_i \sqcup w_{i+1}$ by

$$w_i' = ab \dots c.$$

It follows from the second condition for mergeability that v is still a term in ω' so $(\omega', v) \in X$.

Ex. Suppose $w = abcdefg$ and

$$(\omega, v) = (a \sqcup b \sqcup c \sqcup de \sqcup fg, fcbdgea)$$

Then $w_3 = c$ is mergeable with $w_4 = de$ and

$$\mu(\omega, v) = (a \sqcup b \sqcup cde \sqcup fg, fcbdgea).$$

To define the involution ι, consider (ω, v) and find the smallest index i, if any, such that w_i is either splittable or mergeable. Let

$$\iota(\omega, v) = \begin{cases} \sigma(\omega, v) & \text{if } w_i \text{ is splittable,} \\ \mu(\omega, v) & \text{if } w_i \text{ is mergeable.} \end{cases}$$

If i does not exist, then (ω, v) is a fixed point of ι.

Ex. Suppose $w = abcdefg$ and

$$(\omega, v) = (a \sqcup\!\sqcup b \sqcup\!\sqcup cde \sqcup\!\sqcup fg, fcbdgea).$$

$i \neq 1$ since a is not left of b in v. Similarly $i \neq 2$. But w_3 splits

$$\iota(\omega, v) = (a \sqcup\!\sqcup b \sqcup\!\sqcup c \sqcup\!\sqcup de \sqcup\!\sqcup fg, fcbdgea).$$

The minimality of i makes ι an involution. It is clearly sign reversing. And since v does not change when applying ι, the corresponding terms in Takeuchi's sum will cancel. If (ω, v) is fixed then $|w_1| = \cdots = |w_n| = 1$ because no w_i is splittable. And since no w_i and w_{i+1} are mergeable, we must have

$$v = w_n \ldots w_1 = \operatorname{rev} w. \qquad \square$$

In addition, we have applied the split-merge method in:
(1) The Hopf algebra of polynomials.
(2) The incidence Hopf algebra of graphs (Humpert and Martin).
(3) QSym in the monomial basis.
(4) QSym in the fundamental basis.
(5) mQSym in the fundamental basis (Patrias).

Others have applied this method in:
(6) A Hopf algebra of word complexes (N. Bergeron and Ceballos).
(7) A Hopf algebra of involutions (Dahlberg).
(8) A Hopf algebra of simplicial complexes (Benedetti, Hallam and Machacek).

We have applied other involutions to derive new formulas for particular values of S in:
(9) NSym in the immaculate basis.
(10) The Malvenuto-Reutenauer Hopf algebra of permutations.

In progress:
(11) The Poirier-Reutenauer Hopf algebra of tableaux.

Open Problems

(1) Find merge/split proofs for other antipode formulas, for example in the Hopf algebra of symmetric functions using the Schur basis.

(2) Find a general theorem of the form "If H is a connected graded Hopf algebra having a basis satisfying property X then there is an explicit merge/split involution giving a cancellation-free formula for S."

(3) Generalize the cancellation-free formulas we have found for hooks and 2-rowed compositions in the immaculate basis for NSym to other compositions. A similar question could be asked for the Malvenuto-Reutenauer and Porier-Reutenauer Hopf algebras.

THANKS FOR LISTENING!